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Time duration : 1 h and 30 min

Exam of Mathematics

Exercise 1 (10pt) : In $\mathbb{R}^{3\times3}$, we consider the following matrices :

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 & 0 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

1. Determine the matrices A^T and B^T.
2. Give the lower and upper triagular of A and B.
3. Calculate the determinants of matrices A and B.
4. Find the inverse of the invertible matrix. .
5. Use the matrix method to find x, y and z of this system

$$\begin{cases} x+y & = 2 \\ -2x+z & = 1 \\ x+y+z & = 0 \end{cases} \tag{1}$$

Exercise 2 (6pt) : Let $A = (1,1,-1)$, $B = (-1,0,1)$ and $C = (1,1,0)$ be three points in $\mathbb{R}^3$.

1. (a) Determine $\overrightarrow{AB}$ and $\overrightarrow{AC}$.
   (b) Give $\|\overrightarrow{AB}\|$ and $\|\overrightarrow{AC}\|$
   (c) Calculate $\overrightarrow{AC}.\overrightarrow{AB}$.
   (d) Give $\cos(\overrightarrow{AB}, \overrightarrow{AC})$.
2. Let (P), $-2x - y + 2z + 4 = 0$
   (a) Determine distance $D(C,(P))$.
   (b) Find the pojection point of C on (P).

Exercise 3 (4pt) :
Let f be a function defined as

$$f(x) = \sin(x)\cos(x).$$

Solve the following :

1. Solve $f(x) = 0$ in $\mathbb{R}$.
2. Compute $\int f(x)dx$.
3. Conclude $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} f(x)f(y)dxdy$.

Good Luck!

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Solution of Mathematics

Solution 1

1. The matrices A^T and B^T.

$$A^T = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad B^T = \begin{pmatrix} -1 & -1 & 1 \\ 2 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. The lower and upper triagular of A and B.

$$Upper(A) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad Upper(B) = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$Lower(A) = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad Lower(B) = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

3. The determinants of matrices A and B. For matrix A :

$$\det(A) = 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} -2 & 0 \\ 1 & 1 \end{vmatrix}.$$

The minors are :

$$\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (0)(1) - (1)(1) = -1, \quad \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = (-2)(1) - (1)(1) = -2 - 1 = -3.$$

Substituting back :

$$\det(A) = 1(-1) - 1(-3) + 0(0) = -1 + 3 = 2.$$

For matrix B :

$$\det(B) = -1\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}.$$

The minors are :

$$\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = (2)(1) - (0)(-1) = 2, \quad \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} = (-1)(1) - (0)(1) = -1.$$

Substituting back :

$$\det(B) = -1(2) - 2(-1) + 0 = -2 + 2 = 0.$$

4. **Invertibility :**
   (a) For A, we have $\det(A) = 2 \neq 0$, so A is invertible.
   (b) For B, we have $\det(B) = 0$, so B is not invertible.

   Inverse of A :
   The formula for the inverse of a 3×3 matrix is :

$$A^{-1} = \frac{1}{\det(A)} C_A^T,$$

   where C_A is the cofactor matrix. We compute the cofactors of A as follows :

$$C_{ij} = (-1)^{i+j} \det(M_{ij}),$$

   where M_{ij} is the minor obtained by removing row i and column j.

   (a) $C_{11} = \det\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = (0)(1) - (1)(1) = -1,$

(b) $C_{12} = -\det\begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} = -((-2)(1) - (1)(1)) = -(-2-1) = 3,$

(c) $C_{13} = \det\begin{pmatrix} -2 & 0 \\ 1 & 1 \end{pmatrix} = (-2)(1) - (0)(1) = -2,$

(d) $C_{21} = -\det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = -((1)(1) - (0)(1)) = -1,$

(e) $C_{22} = \det\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = (1)(1) - (0)(1) = 1,$

(f) $C_{23} = -\det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = -((1)(1) - (1)(1)) = 0,$

(g) $C_{31} = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (1)(1) - (0)(0) = 1,$

(h) $C_{32} = -\det\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = -((1)(1) - (-2)(0)) = -(1+0) = -1,$

(i) $C_{33} = \det\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix} = (1)(0) - (1)(-2) = 2.$

The cofactor matrix is :

$$\mathrm{C}_A = \begin{pmatrix} -1 & 3 & -2 \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{pmatrix}.$$

The transpose of the cofactor matrix :

$$C_A^T = \begin{pmatrix} -1 & -1 & 1 \\ 3 & 1 & -1 \\ -2 & 0 & 2 \end{pmatrix}.$$

The determinant of A is $\det(A) = 2$. Therefore, the inverse is :

$$A^{-1} = \frac{1}{2}\begin{pmatrix} -1 & -1 & 1 \\ 3 & 1 & -1 \\ -2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \end{pmatrix}.$$

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5. The solution is given by $\mathbf{X} = A^{-1}\mathbf{b}$. Solving for $\mathbf{X}$, we find :

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} \\ \frac{7}{2} \\ -2 \end{pmatrix}.$$

Solution 2 :
Let $A = (1,1,-1)$, $B = (-1,0,1)$, and $C = (1,1,0)$ be three points in $\mathbb{R}^3$.

1. (a) **Determine** $\overrightarrow{AB}$ **and** $\overrightarrow{AC}$:

$$\overrightarrow{AB} = B - A = (-1-1, 0-1, 1-(-1)) = (-2,-1,2),$$
$$\overrightarrow{AC} = C - A = (1-1, 1-1, 0-(-1)) = (0,0,1).$$

(b) **Calculate** $\|\overrightarrow{AB}\|$ **and** $\|\overrightarrow{AC}\|$:

$$\|\overrightarrow{AB}\| = \sqrt{(-2)^2 + (-1)^2 + 2^2} = \sqrt{4+1+4} = \sqrt{9} = 3,$$
$$\|\overrightarrow{AC}\| = \sqrt{(0)^2 + (0)^2 + (1)^2} = \sqrt{1} = 1.$$

(c) **Calculate** $\overrightarrow{AC} \cdot \overrightarrow{AB}$ **and give** $\cos(\theta)$:

$$\overrightarrow{AC} \cdot \overrightarrow{AB} = (0)(-2) + (0)(-1) + (1)(2) = 0 + 0 + 2 = 2.$$

Using the formula $\cos(\theta) = \frac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{\|\overrightarrow{AC}\|\|\overrightarrow{AB}\|}$:

$$\cos(\theta) = \frac{2}{(1)(3)} = \frac{2}{3}.$$

2. Let $(P): -2x - y + 2z + 4 = 0$.
   (a) **Determine the distance** $D(C,(P))$: The formula for the distance from a point $C = (x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$ is :
   $$D(C,(P)) = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$
   Vector $v = (a, b, c) = (-2, -1, 2)$. Substituting $a = -2$, $b = -1$, $c = 2$, $d = 4$, and $C = (1, 1, 0)$:
   $$D(C,(P)) = \frac{|(-2)(1) + (-1)(1) + (2)(0) + 4|}{\sqrt{(-2)^2 + (-1)^2 + 2^2}} = \frac{|-2-1+4|}{\sqrt{4+1+4}} = \frac{|1|}{\sqrt{9}} = \frac{1}{3}.$$
   (b) **Find the projection point of** C **on** (P) : First, we define the line equation
   $$\begin{cases} x &= -2t + 1 \\ y &= -t + 1 \\ z &= 2t \end{cases} \tag{1}$$
   By substituting :
   $$-2x - y + 2z + 4 = (-2)(-2t + 1) + (-1)(-t + 1) + (2)(2t) + 4 = 4t - 2 + t - 1 + 4t + 4 = 0,$$
   then
   $$t = -\frac{1}{9}.$$
   The projection point is :
   $$\left(\frac{11}{9}, \frac{10}{9}, -\frac{2}{9}\right).$$

Solution 3 (4pt) :
Let $f(x) = \sin(x)\cos(x)$. Solve the following :

1. **Solve** $f(x) = 0$ in $[0, 2\pi]$:
   $$f(x) = \sin(x)\cos(x) = 0.$$
   Using the identity $\sin(2x) = 2\sin(x)\cos(x)$, we rewrite the equation as :
   $$\sin(2x) = 0.$$
   The solutions to $\sin(2x) = 0$ in the interval $[0, 2\pi]$ are :
   $$2x = n\pi, \quad n \in \mathbb{Z}.$$
   In the interval $[0, 2\pi]$, this gives :
   $$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi.$$
   Thus, the solutions are $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.
2. **Compute** $\int f(x)\,dx$:
   $$\int f(x)\,dx = \int \sin(x)\cos(x)\,dx.$$
   Using the identity $\sin(2x) = 2\sin(x)\cos(x)$, we rewrite the integral :
   $$\int \sin(x)\cos(x)\,dx = \frac{1}{2}\int \sin(2x)\,dx.$$
   The integral of $\sin(2x)$ is :
   $$\int \sin(2x)\,dx = -\frac{1}{2}\cos(2x).$$
   Therefore,
   $$\int f(x)\,dx = \frac{1}{2}\left(-\frac{1}{2}\cos(2x)\right) = -\frac{1}{4}\cos(2x) + C,$$
   where C is the constant of integration.

3. **Compute** $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} f(x)f(y)\,dx\,dy$: We want to compute the double integral :

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} f(x)f(y)\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\sin(y)\cos(y)\,dx\,dy.$$

The integrand can be factored :

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\sin(y)\cos(y)\,dx\,dy = \left(\int \sin(x)\cos(x)\,dx\right)\left(\int \sin(y)\cos(y)\,dy\right).$$

We already know the solution to $\int \sin(x)\cos(x)\,dx$, which is :

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\,dx = -\frac{1}{4}[\cos(2x)]_0^{\frac{\pi}{4}} = -\frac{1}{4}[\cos(\pi)-\cos(0)] = -\frac{1}{4}[-1-1] = \frac{1}{2}.$$

Thus, the double integral becomes :

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} f(x)f(y)\,dx\,dy = \frac{1}{4}.$$